Onsager-Machlup functional for SLE loop measures

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Onsager-Machlup functionals

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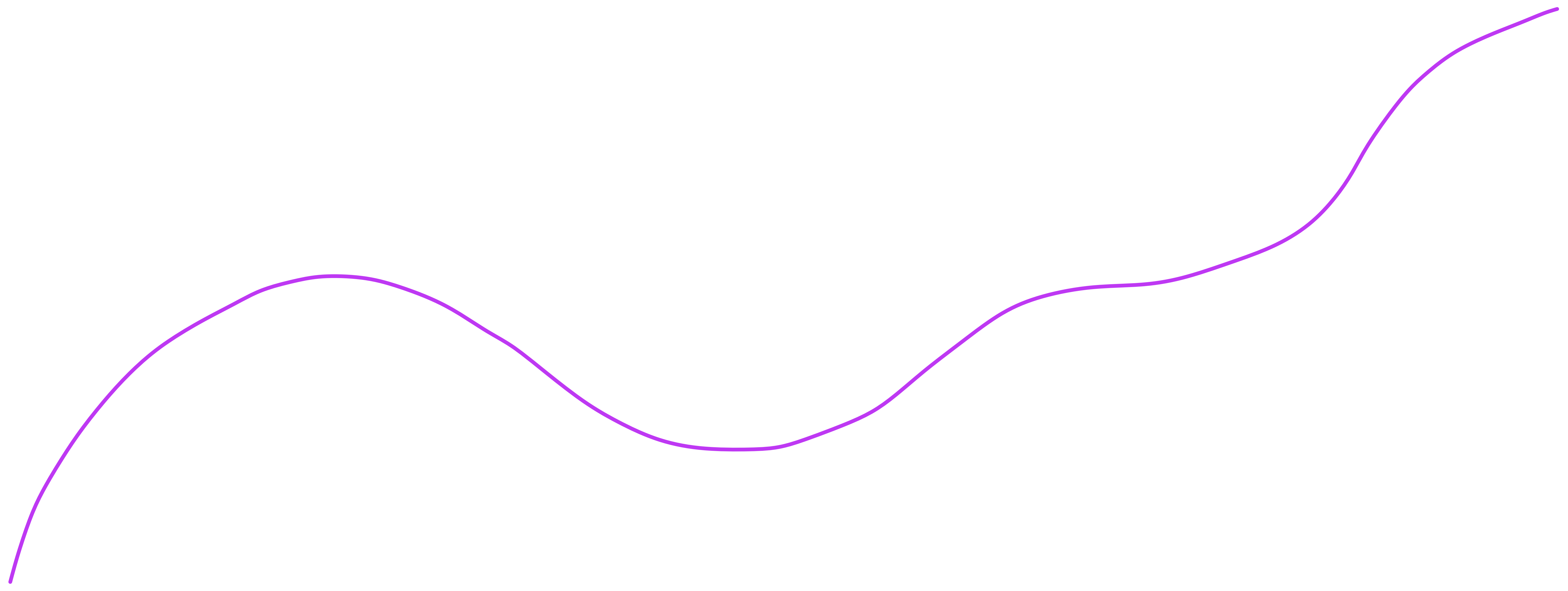

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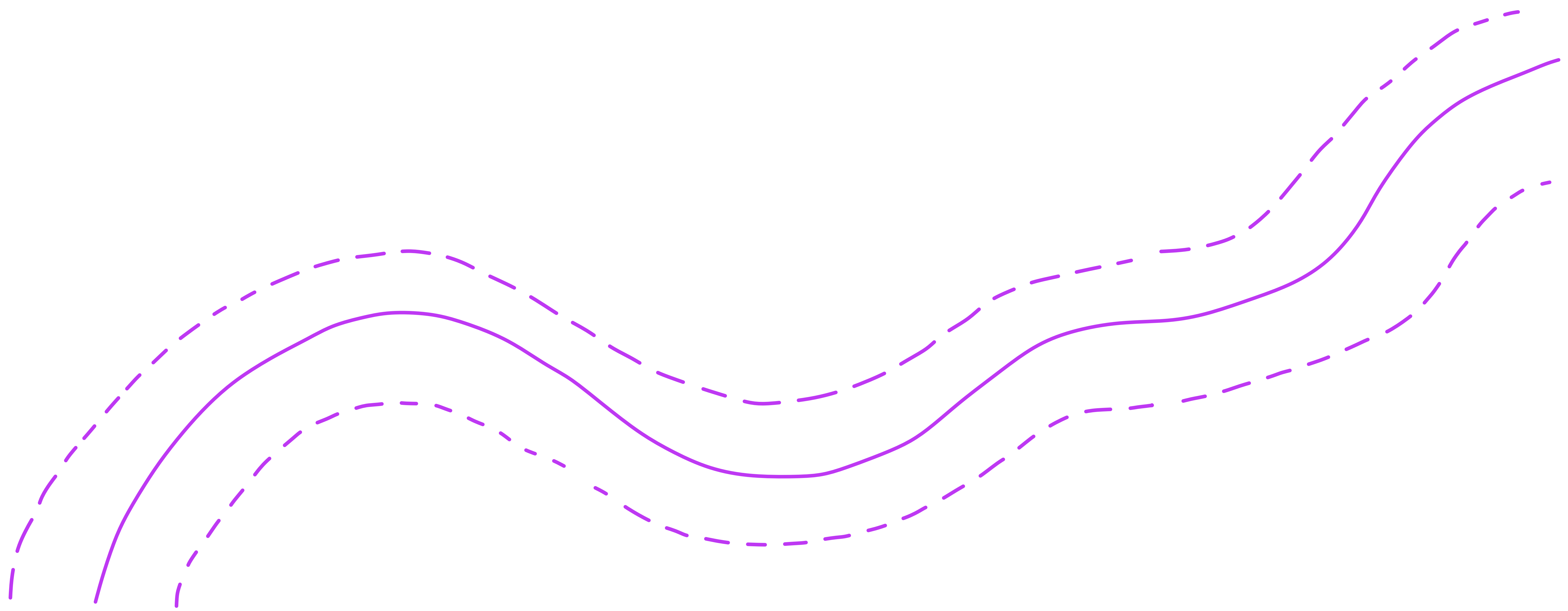

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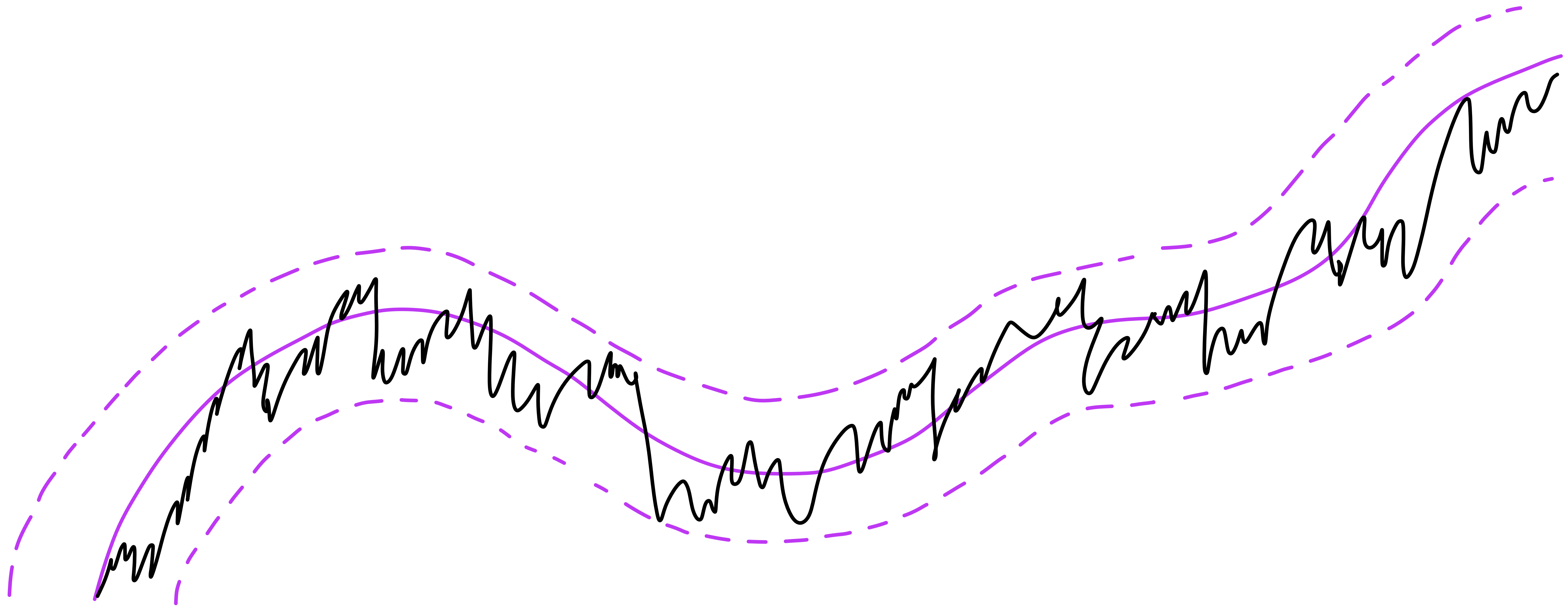

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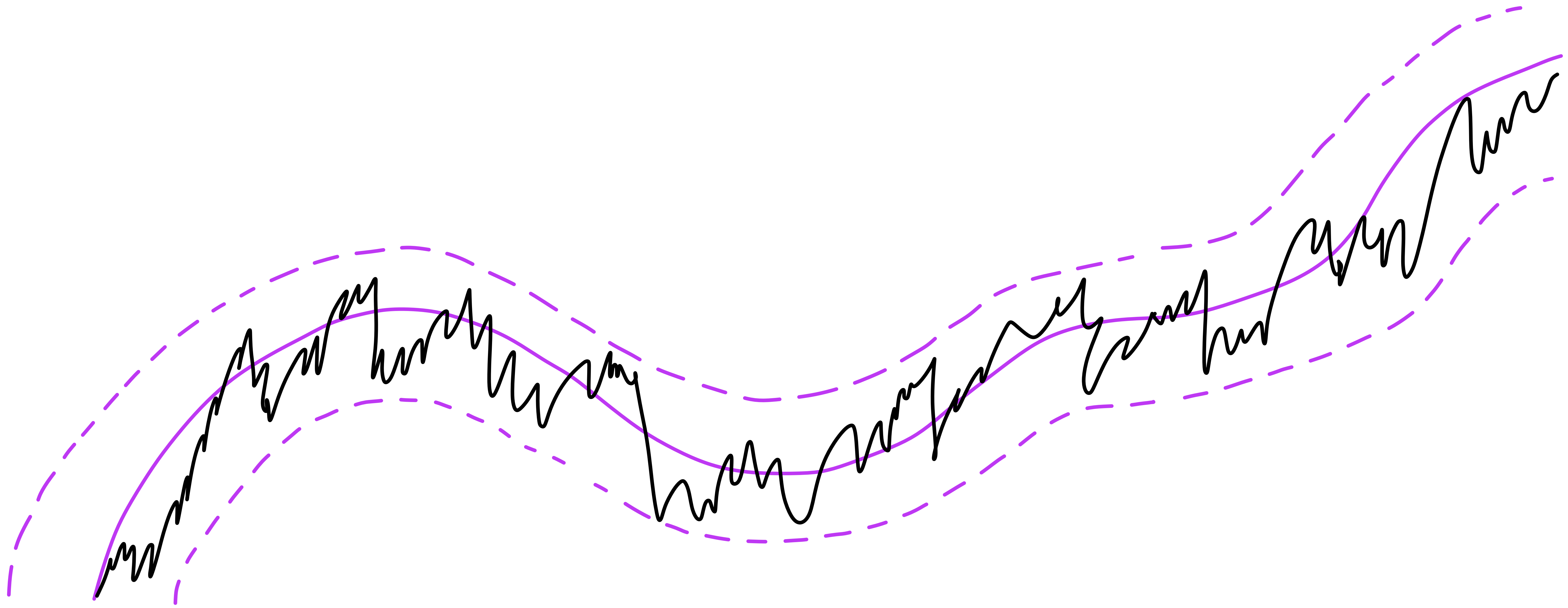

$$\frac{\mathbb{P}\left(\max_{0\leqslant s\leqslant 1}|B_s-\phi(s)|<\varepsilon\right)}{\mathbb{P}\left(\max_{0\leqslant s\leqslant 1}|B_s|<\varepsilon\right)}$$

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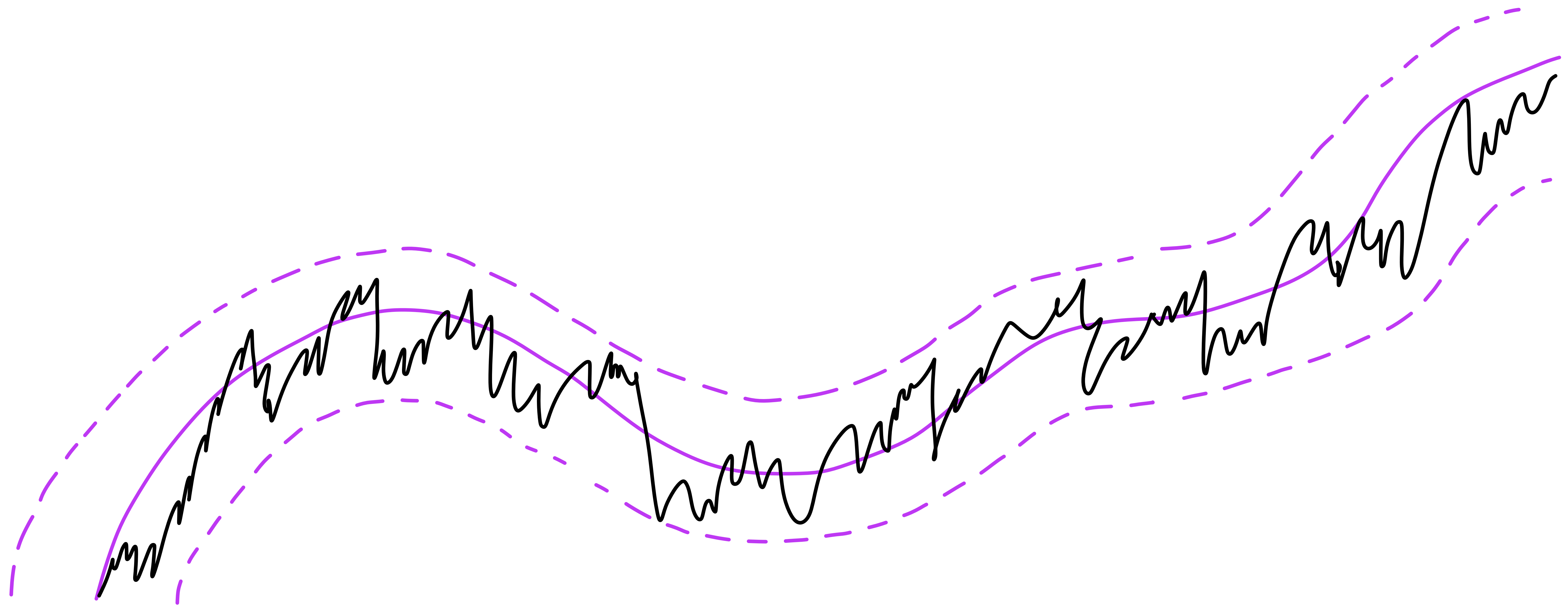

Theorem (Onsager-Machlup '53)

$$\frac{\mathbb{P}\left(\max_{0\leqslant s\leqslant 1}|B_s-\phi(s)|<\varepsilon\right)}{\mathbb{P}\left(\max_{0\leqslant s\leqslant 1}|B_s|<\varepsilon\right)} \longrightarrow \exp\left(\mathcal{O}(\phi)\right) \qquad \mathcal{O}(\phi) := -\frac{1}{2}\int_0^1 \phi'(t)^2 dt$$

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Let $X \sim \mathcal{N}(0,1)$, then $\mathbb{P}(X \in A) = \int_A \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} ds$

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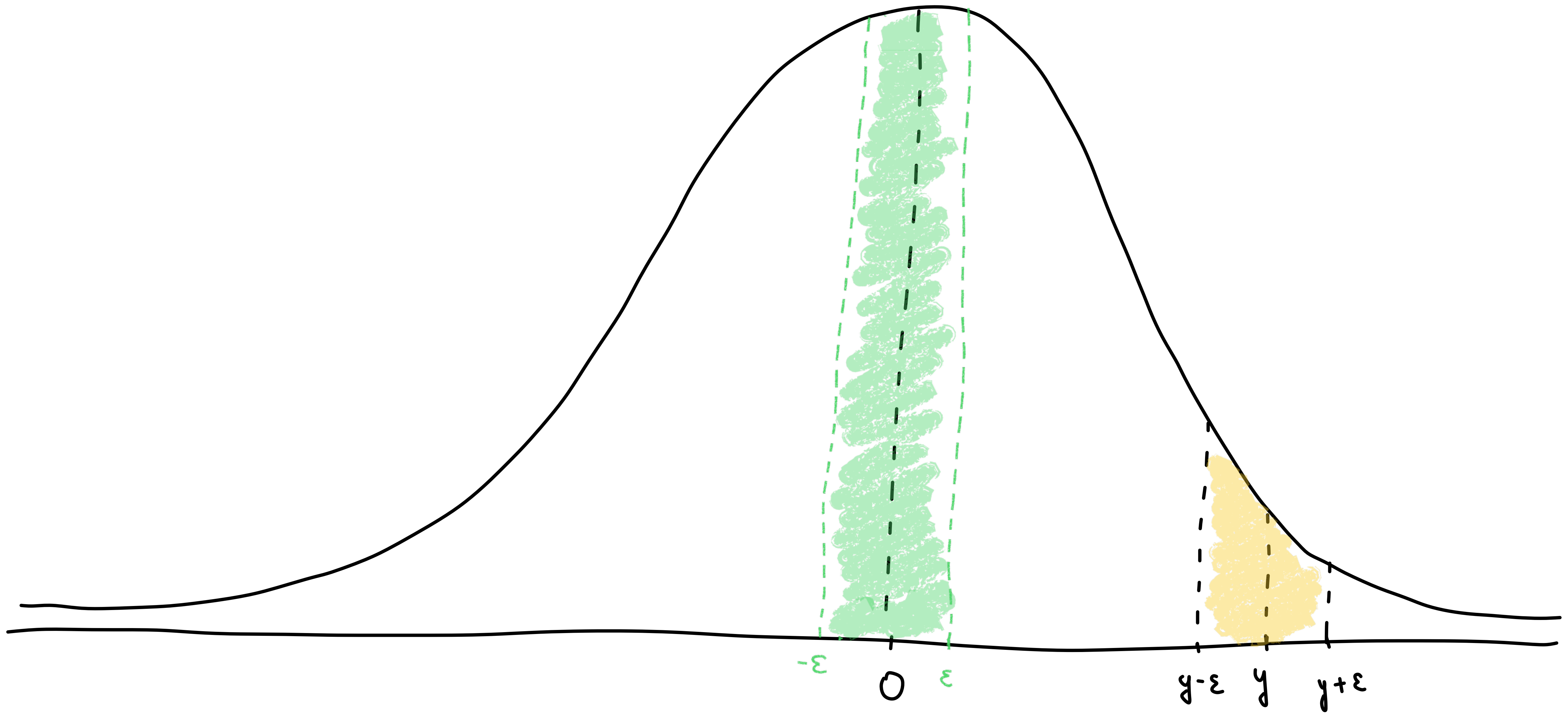

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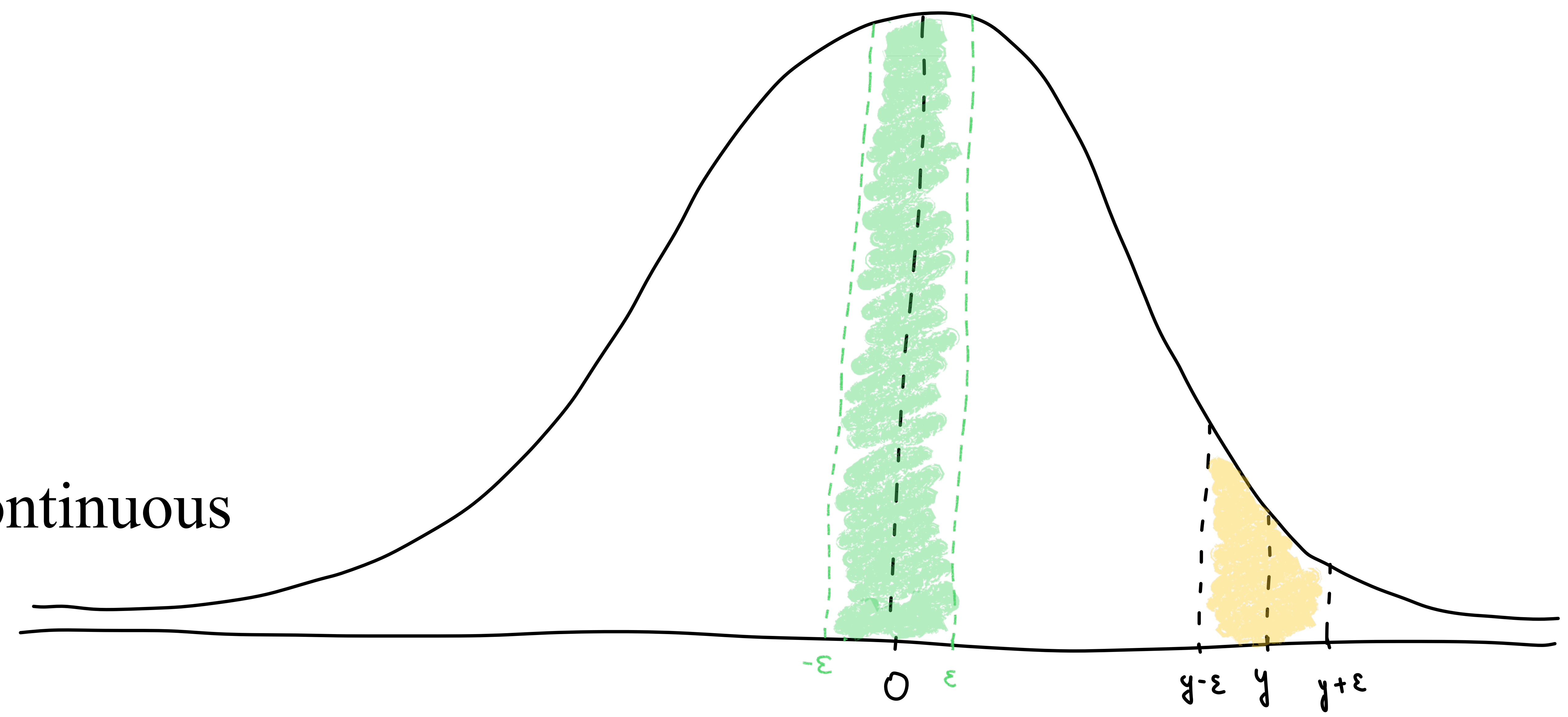

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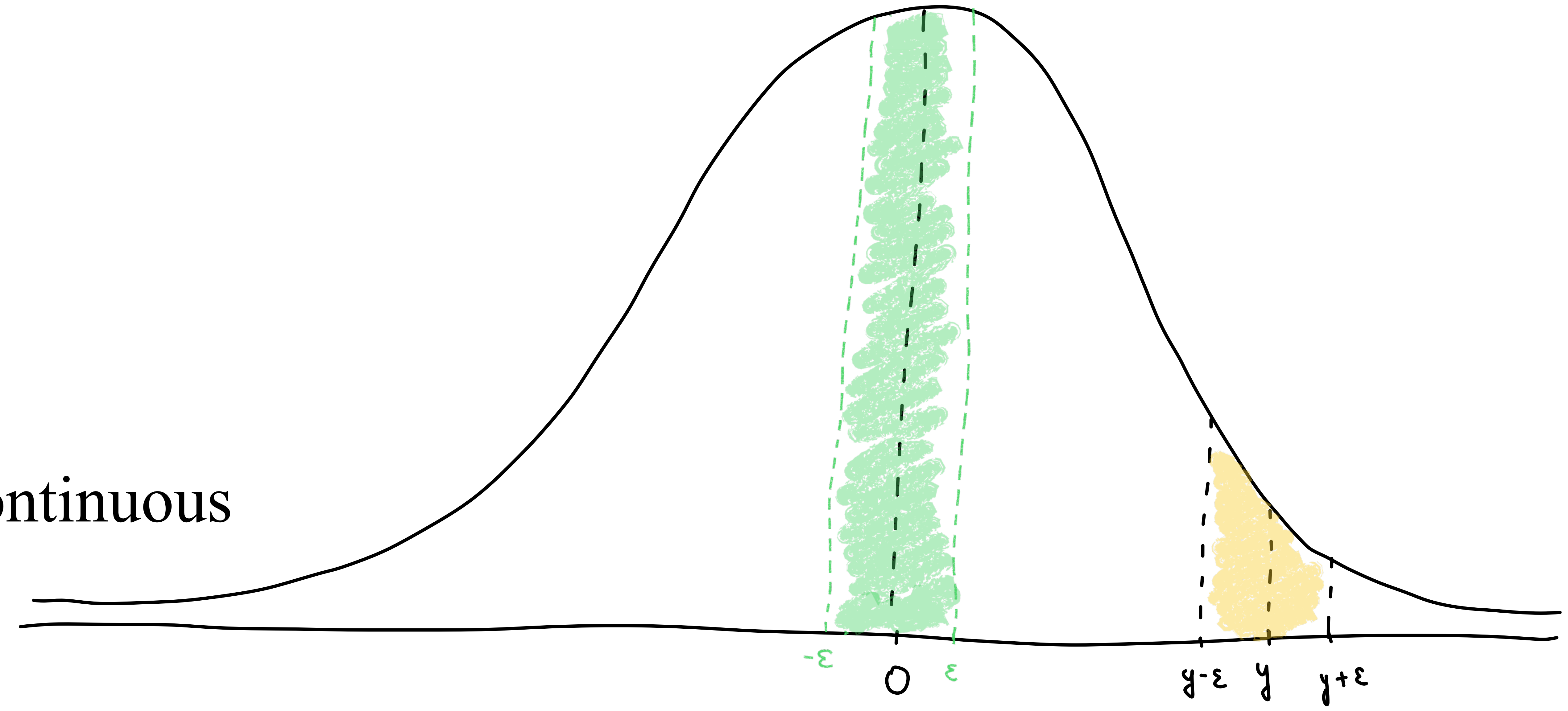

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$(C_0(\mathbb{R}^n), d, \nu)$ metric measure space

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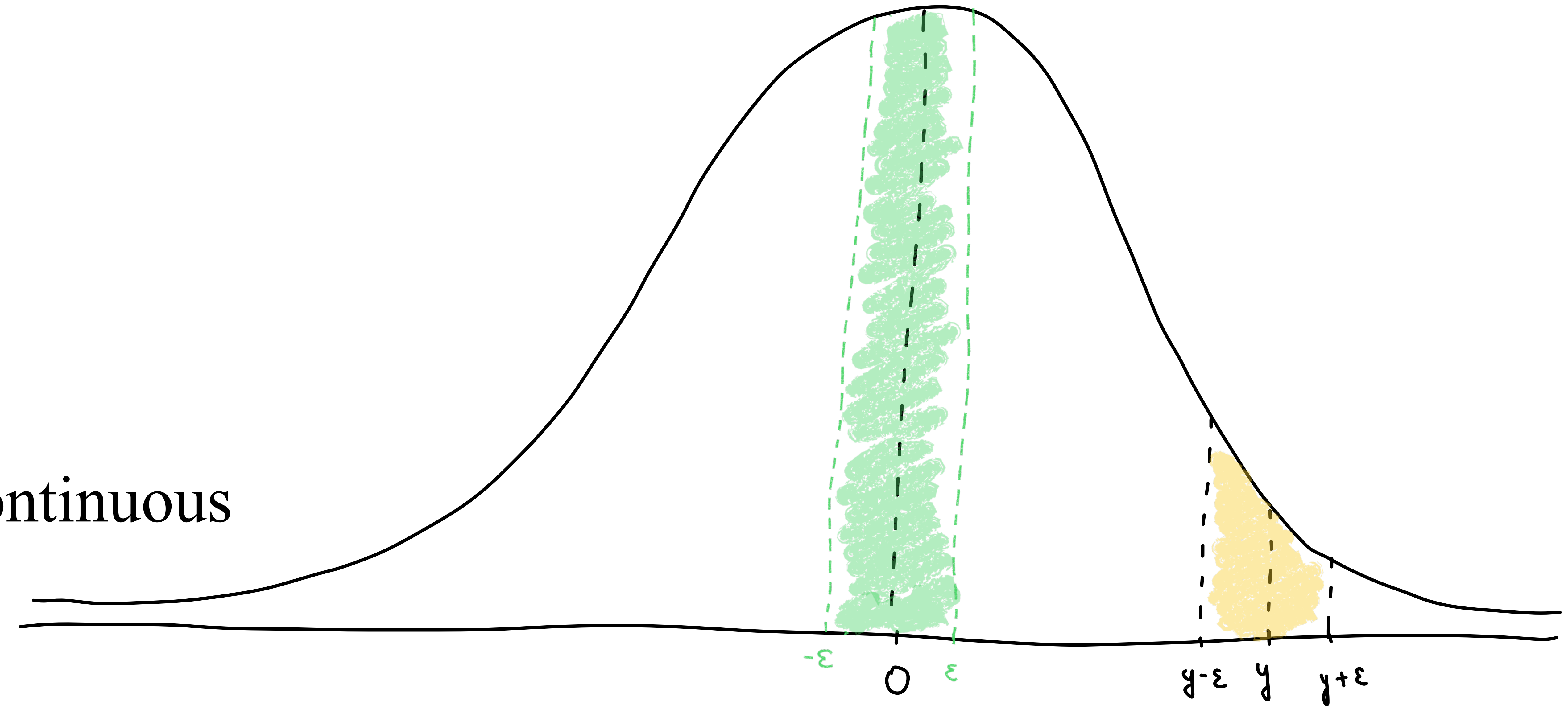

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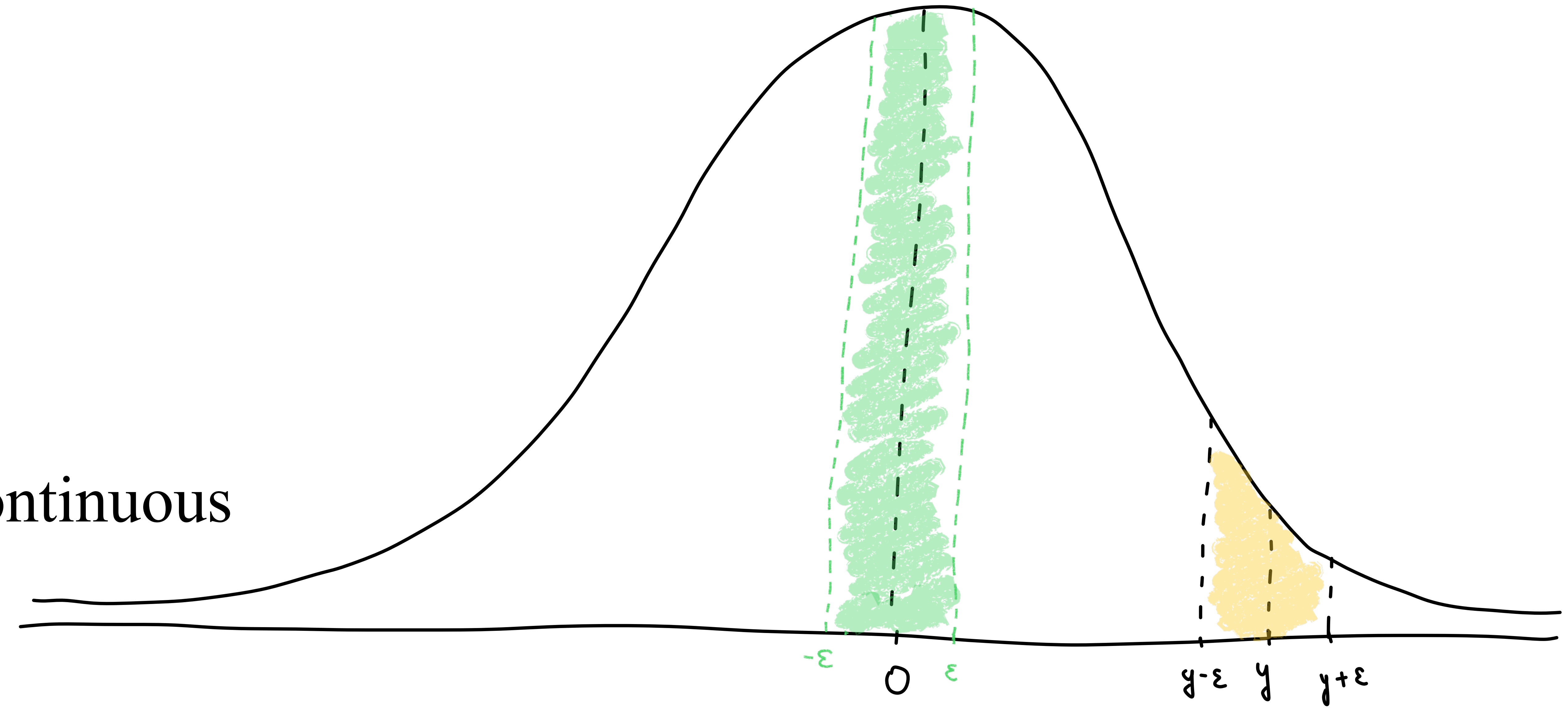

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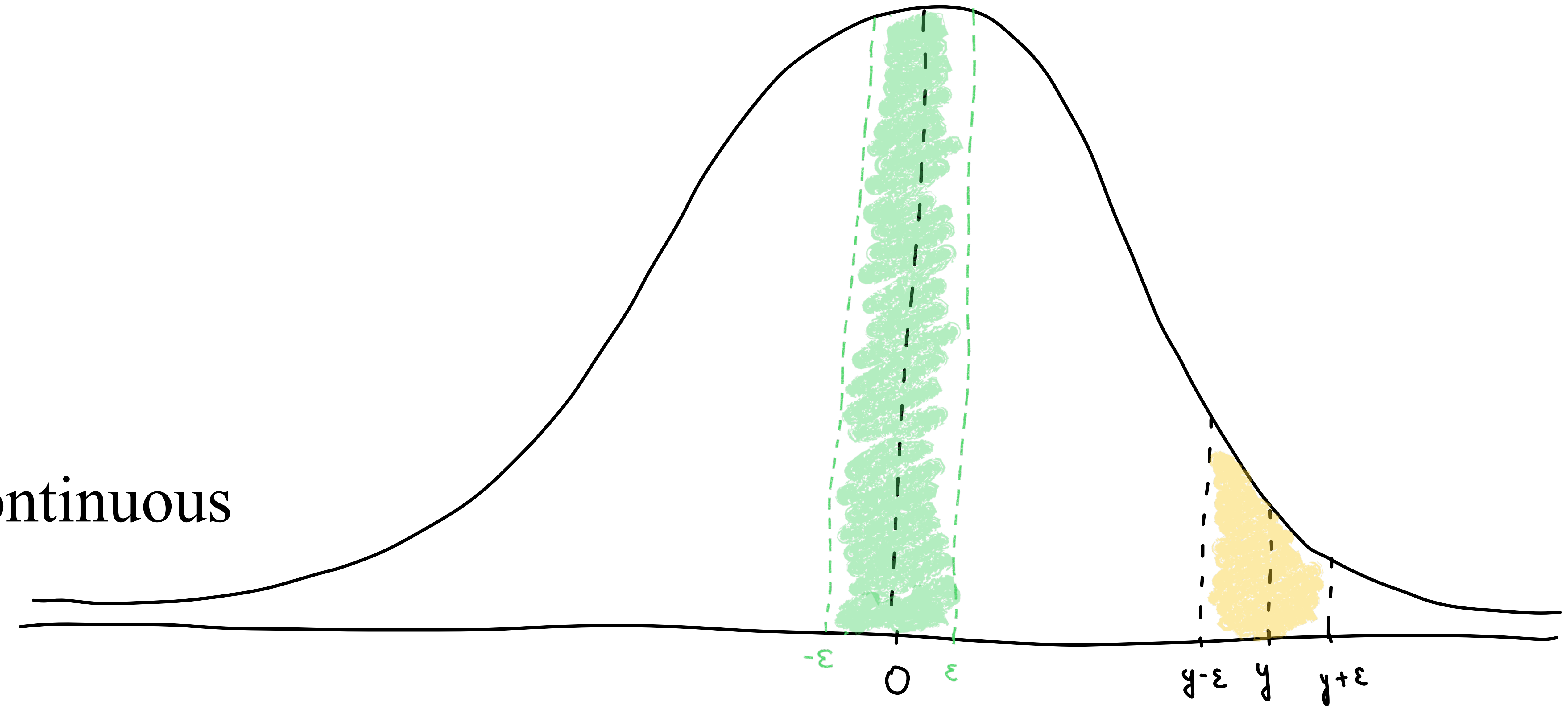

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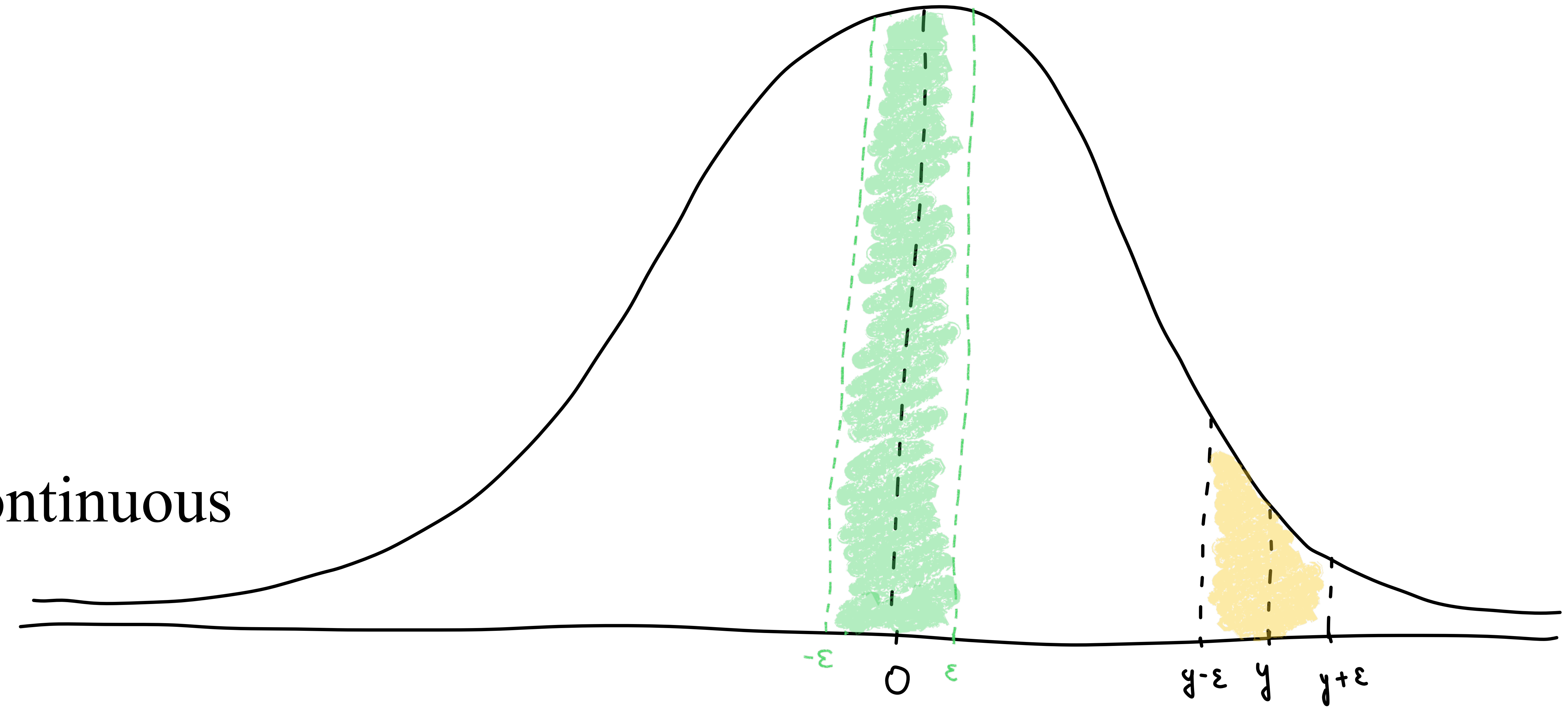

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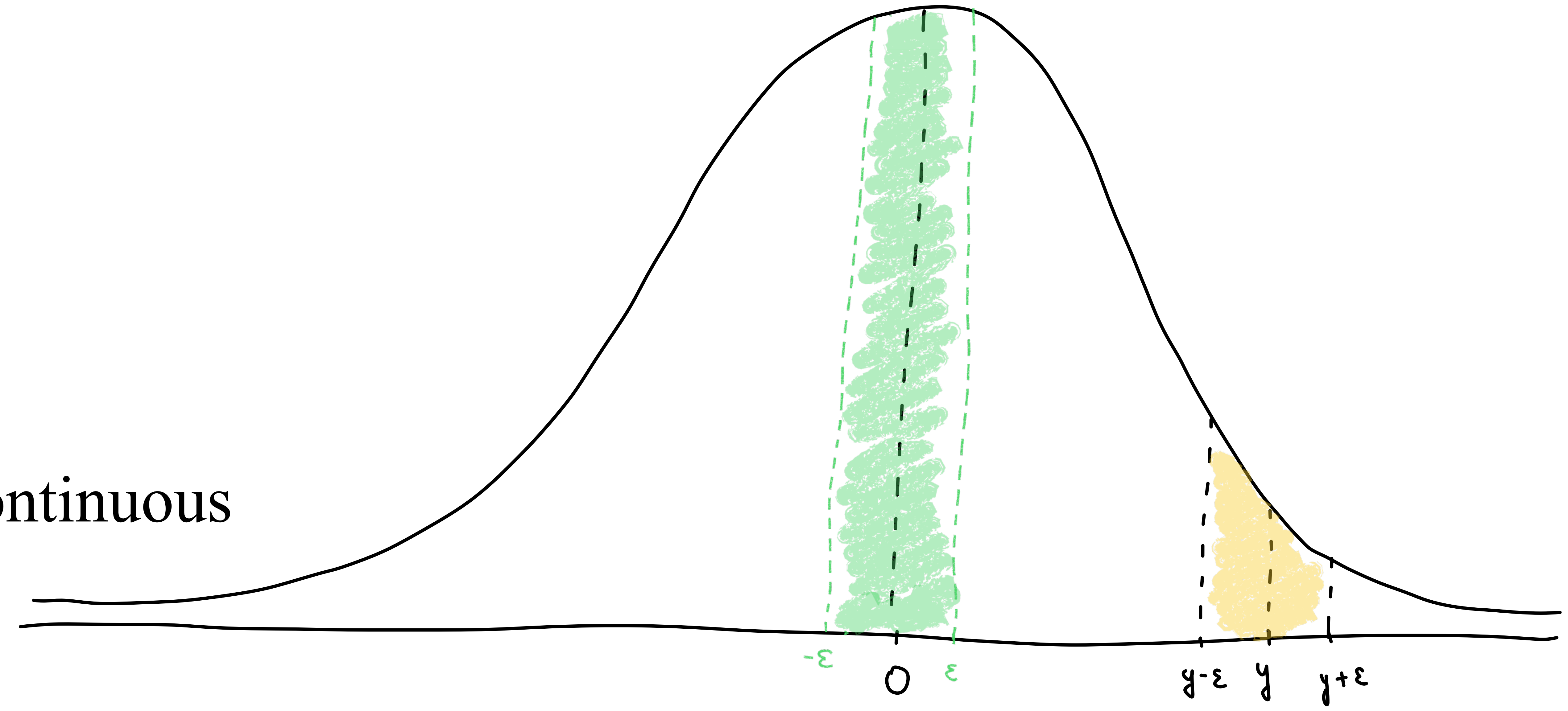

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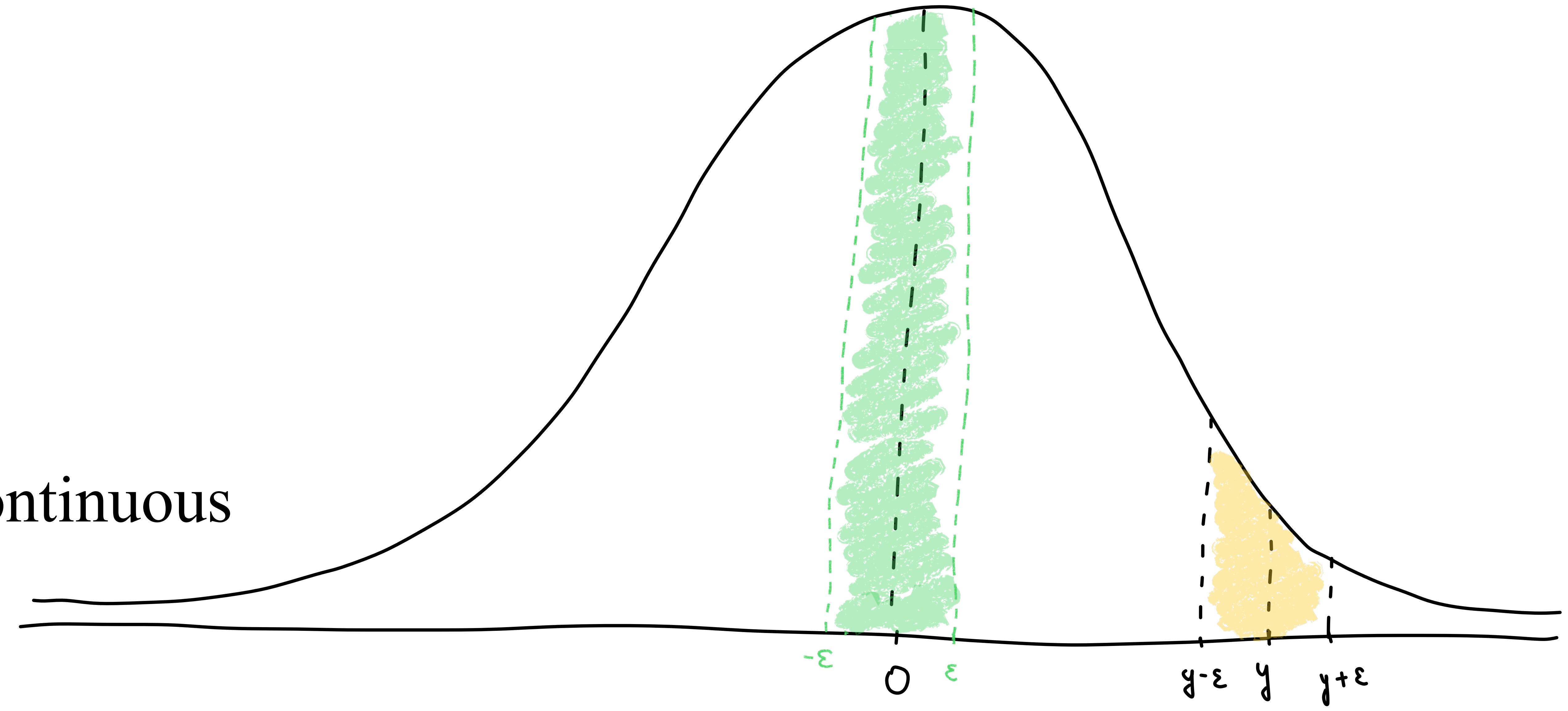

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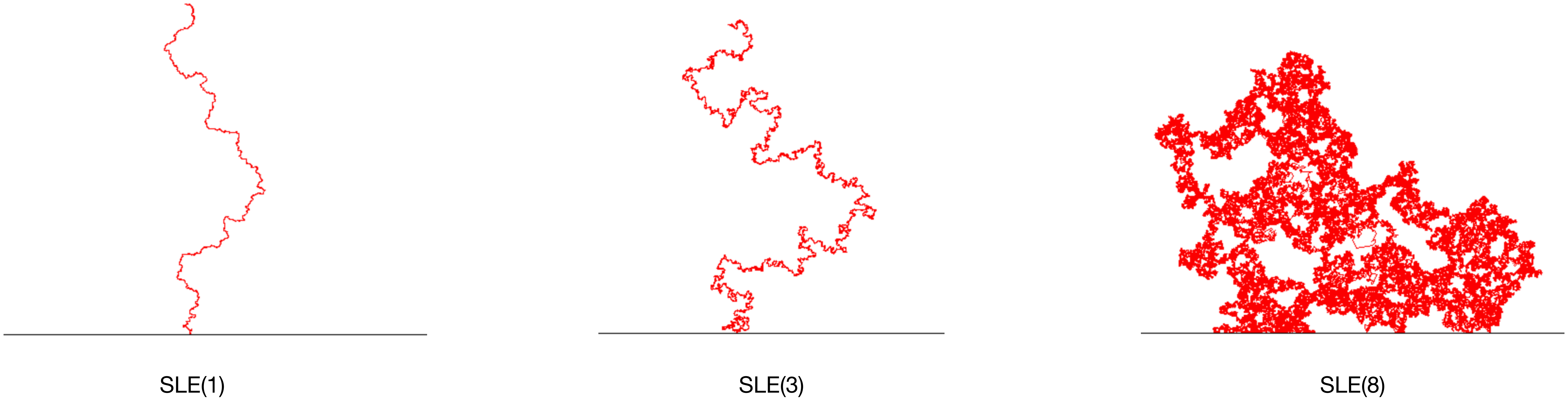
SLE$_\kappa$ loop measures

SLEκ: random curves on the plane $\longrightarrow$ Family of measures on the space of curves

Scaling limit of 2D lattice models: Lawler, Rhode, Schramm, Sheffield, Smirnov, Wang, Werner, Wu, Zhan *et al.*



Courtesy of Tom Kennedy

SLEκ loops: random simple loops on the plane, $\kappa \leq 4$ $\longrightarrow$ Family of measures on the space of simple loops

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Theorem (C. -Wang '23) Let $\kappa \leq 4$ and μ^κ be the SLE$_\kappa$ loop measure. For any analytic simple loop γ in the Riemann sphere we have that

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where $c(\kappa) := (6-\kappa)(3\kappa-8)/2\kappa$ is the central charge of SLE$_\kappa$ and $I^L(\gamma)$ is the Loewner energy of γ .

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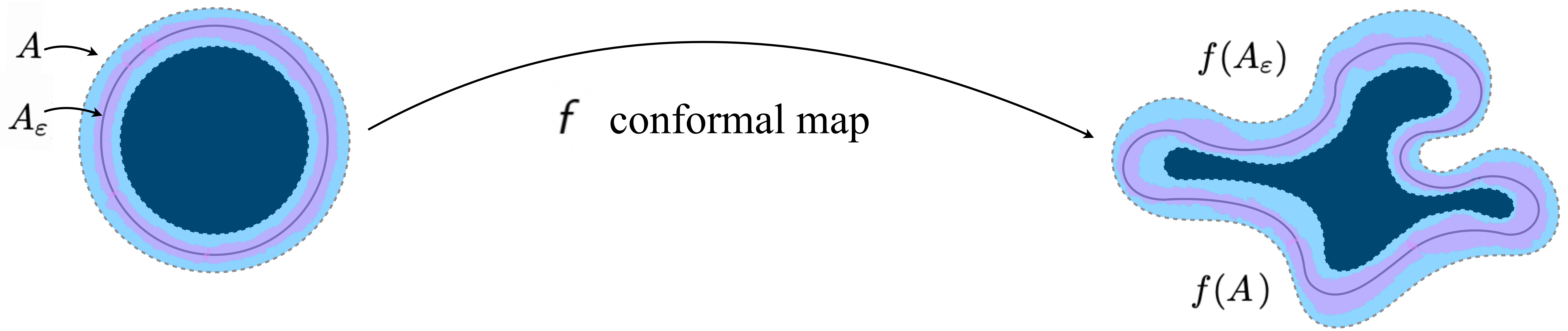

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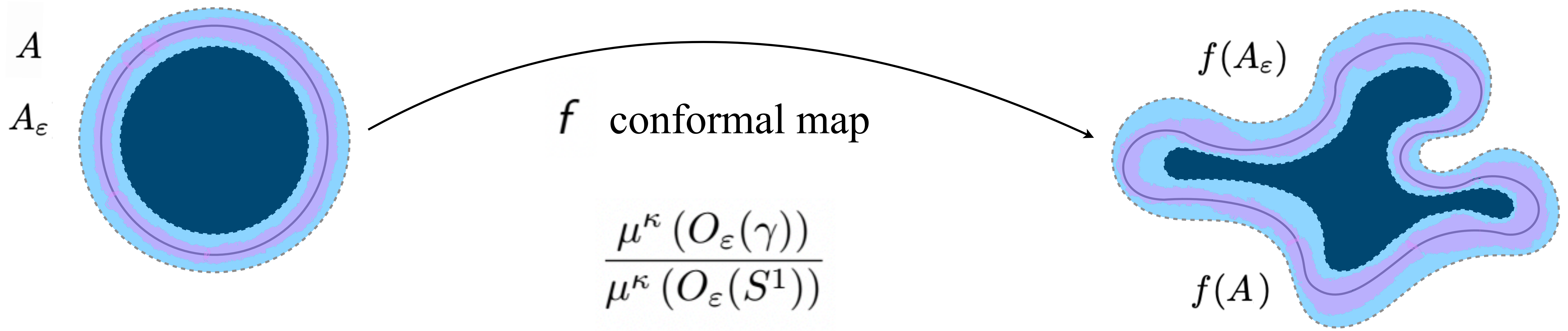

$$\frac{\mu^\kappa(O_\varepsilon(\gamma))}{\mu^\kappa(O_\varepsilon(S^1))}$$

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Basis for the Hausdorff topology (on simple loops)!

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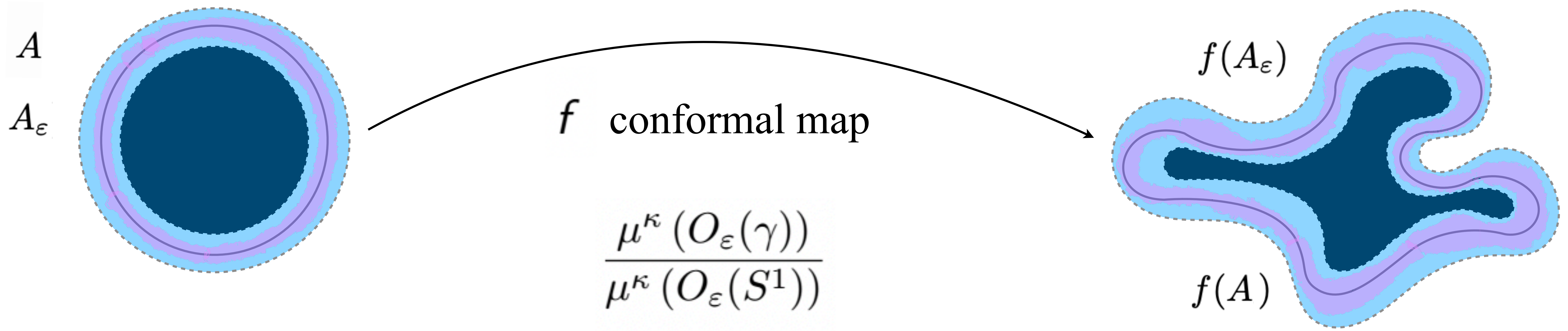

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First define energy of chordal SLEκ using Loenwer theory (Friz-Shekhar '17,Wang '19)

Extension to simple loops on the sphere (Rohde-Wang '21)

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$I^L(\gamma) = 0$ iff γ = circle

$I^L(\gamma)$ is finite iff γ = quasi-circle

Sketch of the proof

Conformal restriction property of μ^κ [Lawler-Schramm-Werner] [Kontsevich-Suhov][Zhan]

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$$\frac{df_*\left(\mu^\kappa 1_{\{\,\cdot\,\subset A\}}\right)}{d\mu^\kappa 1_{\{\,\cdot\,\subset \tilde{A}\}}}\Big|_\eta = \exp\left(\frac{c}{2}\left(\Lambda^*(\eta, \tilde{A}^\varepsilon) - \Lambda^*(f^{-1}(\eta), A^\varepsilon)\right)\right)$$

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$$\Lambda^*(V_1,V_2) = \lim_{R\to\infty}\left(\mathcal{B}(V_1,V_2;\mathbb{D}_R) - \log\log R\right)$$

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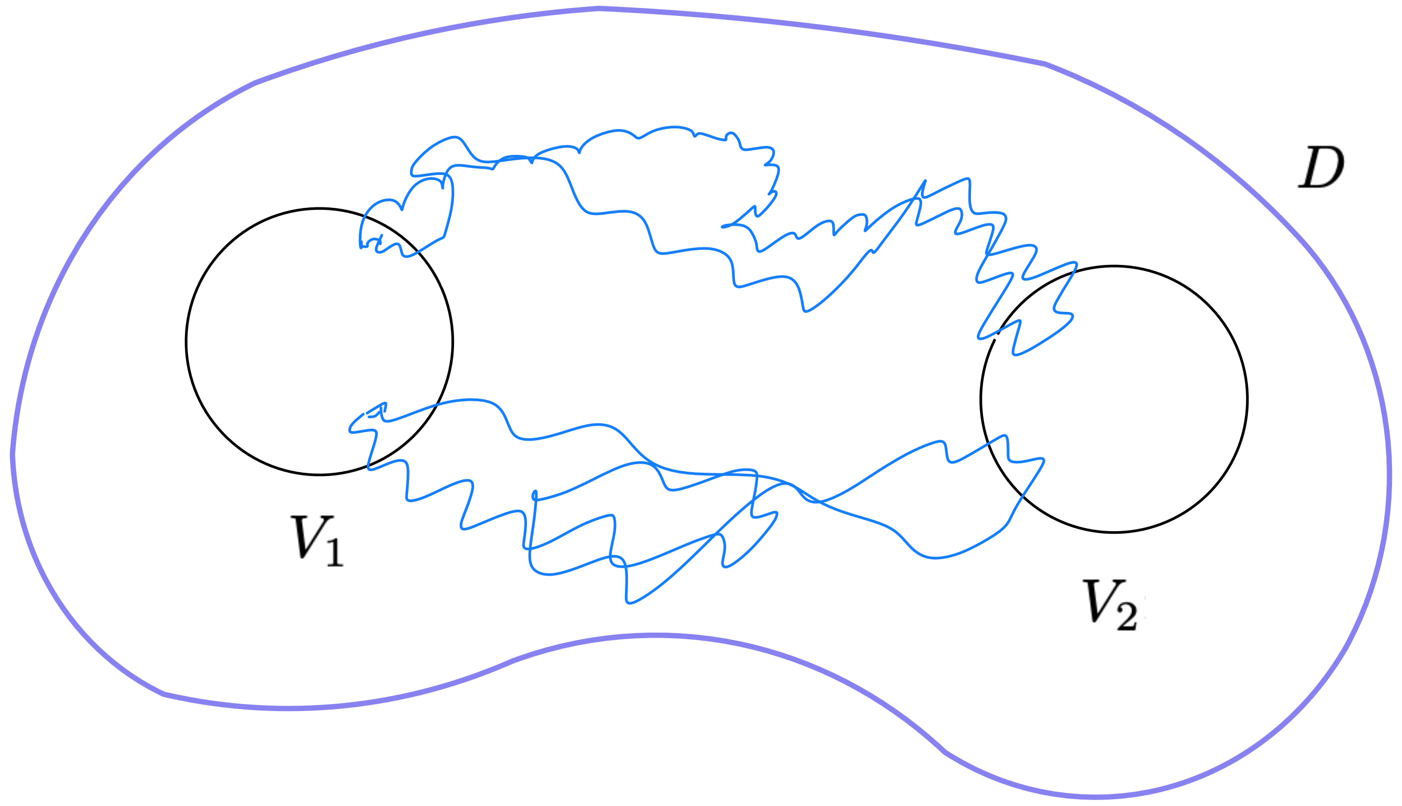

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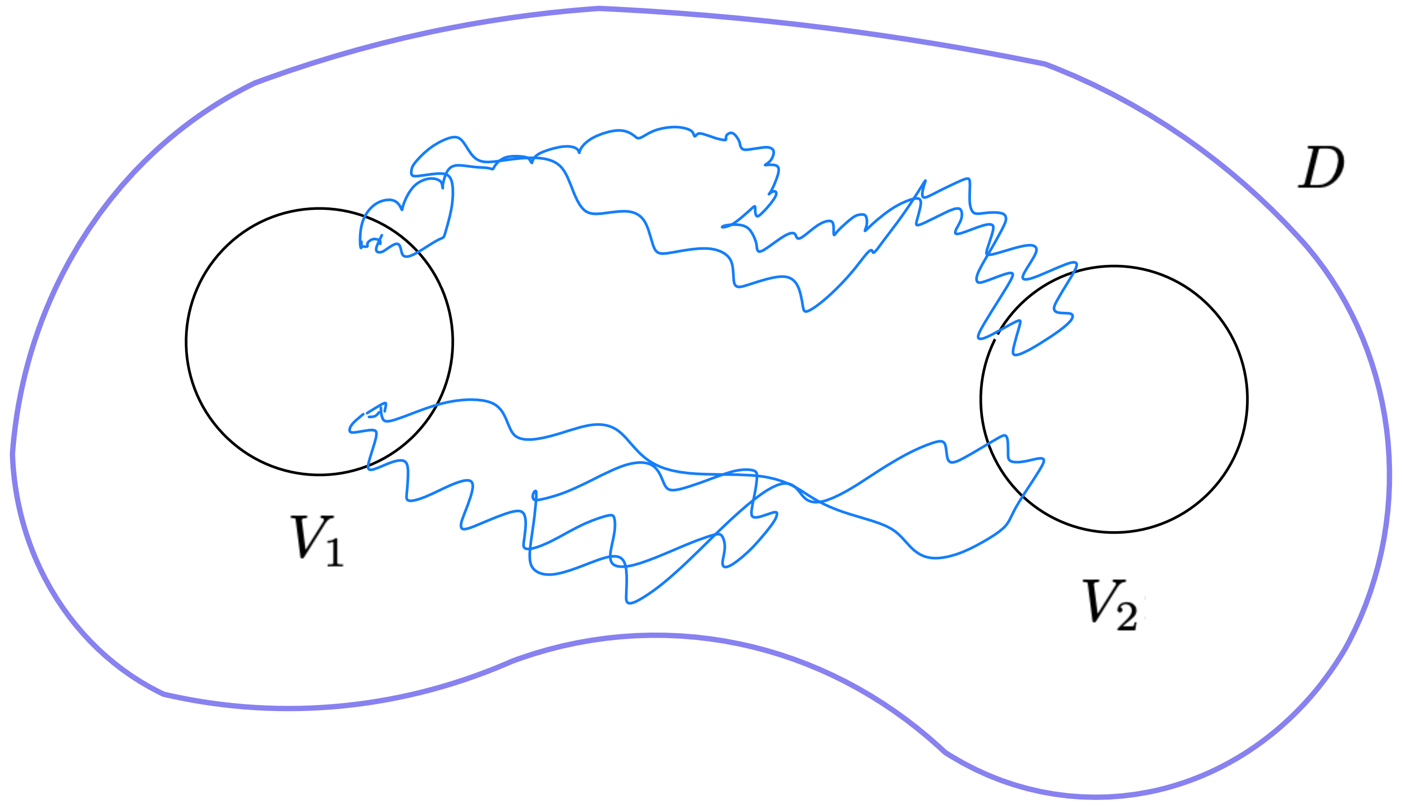

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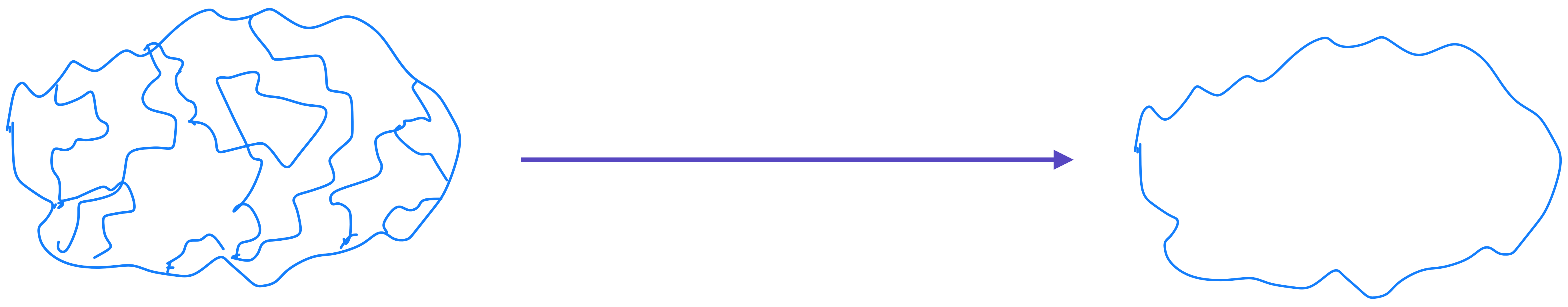
$\mathcal{W}$: Werner measure

Brownian loop δ, not simple

Outer boundary $\partial\delta$, simple



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Löwner energy?

Theorem (Wang '21) $I^L(f(\gamma)) - I^L(\gamma) = 12\mathcal{W}(\gamma, A^c) - 12\mathcal{W}(f(\gamma), f(A)^c)$

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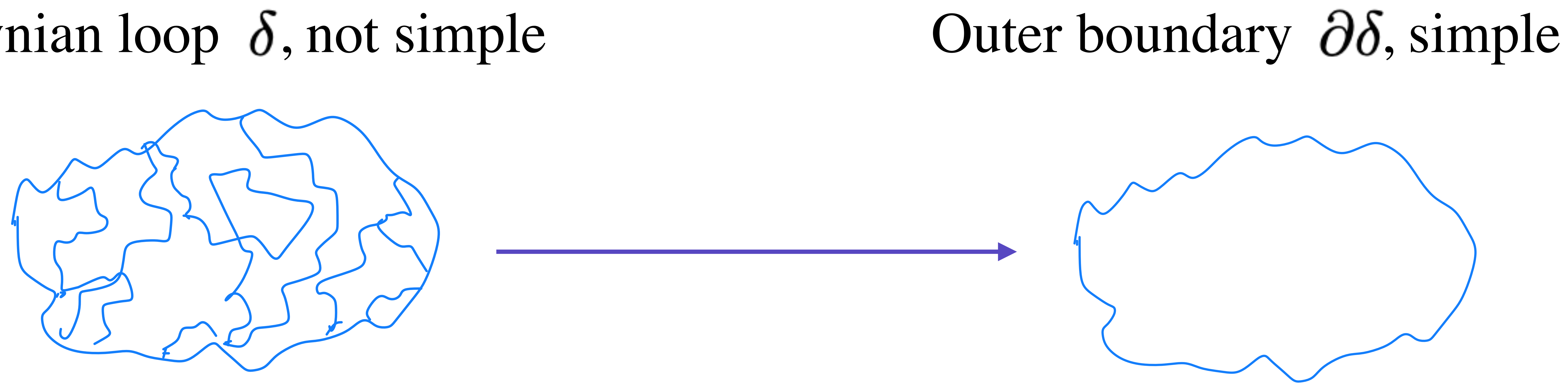

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Remark: $\mathcal{W}(V_1, V_2)$ and $\Lambda^*(V_1, V_2)$ are **different!**

SLE$_\kappa$ Loop Measures

Process	Brownian motion	SLE$_\kappa$ loops
Space	$C_0(\mathbb{R}^n)$ = pointed paths	$\mathcal{SL}(\widehat{\mathbb{C}})$ = simple loops
Measure	ν Wiener measure	μ^κ SLE$_\kappa$ loop measure
Transformation	shift $T_\phi : C_0(\mathbb{R}^n) \longrightarrow C_0(\mathbb{R}^n)$ ϕ fixed, $\eta \to \eta + \phi$, $\nu \to \nu_\phi$	conformal map $f : \mathcal{SL} \longrightarrow \mathcal{SL}$ γ fixed, $S^1 \to \gamma$, $\mu^\kappa \to \mu^\kappa_f$
Functional	$\mathcal{O}(\phi)$, $\dfrac{\nu_\phi(D_\varepsilon(0))}{\nu(D_\varepsilon(0))} \longrightarrow \exp(\mathcal{O}(\phi))$	$I^L(\gamma)$, $\dfrac{\mu^\kappa_f(O_\varepsilon(S^1))}{\mu^\kappa(O_\varepsilon(S^1))} \longrightarrow \exp\left(\frac{c(\kappa)}{24} I^L(\gamma)\right)$